Uniform convolution

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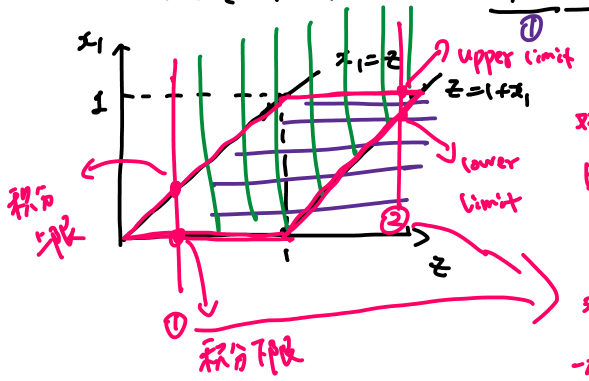
$X_1 \sim \text{Unif}(0,1) \quad X_2 \sim \text{Unif}(0,1)$

$Z = X_1 + X_2$?

$$f_Z(z) = f_{X_1,X_2}(x_1+x_2) = \int_{x_1} f_{X_1,X_2}(x_1, z-x_1)\,dx_1$$

$$= \int_{x_1} f_{X_1}(x_1)\, f_{X_2}(z-x_1)\,dx_1$$

其中 $0 < x_1 < 1$, $0 < z - x_1 < 1$ $\Rightarrow \begin{cases} 0 < x_1 < 1 \\ \underbrace{x_1 < z}_{①} \underbrace{< 1 + x_1}_{②} \end{cases}$



对x_1积分方向↑
因为x_1在y轴上.

对于①,②积分上下限不一样. ∴分段求density.

① $\int_0^z 1\,dx_1 = z \qquad 0 < z < 1$

② $\int_{z-1}^1 1\,dx_1 = x_1 \Big|_{z-1}^1 = 1 - (z-1) = 2 - z \qquad 1 < z < 2$

$$\Rightarrow f_Z(z) = \begin{cases} z & 0 < z < 1 \\ 2 - z & 1 < z < 2 \end{cases}$$

if $X_1 \sim \text{Unif}(\theta, \theta+1)$; $X_2 \sim \text{Unif}(\theta, \theta+1)$

$Y_1 \sim \text{Unif}(0,1)$; $Y_2 \sim \text{Unif}(0,1)$

$X_1 = Y_1 + \theta \qquad X_2 = Y_2 + \theta$

$\therefore X_1 + X_2 = Y_1 + Y_2 + 2\theta$

$Z' = Z + 2\theta \Rightarrow \quad Z = Z' - 2\theta$ transformation

$$\therefore \underbrace{f(z')}_{8.13(b)} = f_Z(z') = \begin{cases} z' - 2\theta & 0 < z' - 2\theta < 1 \\ 2 - z' + 2\theta & 1 < z' - 2\theta < 2 \end{cases}$$

$$= \begin{cases} z' - 2\theta & 2\theta < z' < 2\theta + 1 \\ 2 - z' + 2\theta & \cancel{1+2\theta} < z' < 2\theta + 2 \end{cases}$$